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The stability of the Couette flow of helium II

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The stability of Couette flow in HeII is considered by an analysis of the HVBK equations. These equations are based on the Landau two-fluid model of HeII and include mutual friction between the normal and superfluid components, and the vortex tension due to the presence of superfluid vortices. We find that the vortex tension strongly affects the nature of the Taylor instability at temperatures below ≈ 2.05 K. The effect of the vortex tension is to make non-axisymmetric modes the most unstable, and to make the critical axial wavelength very long.

We compare our results with experiments.

1. Introduction

The flow of a liquid between rotating concentric cylinders, known as Couette flow, has long been recognized as one of the fundamental problems in fluid dynamics. Couette flow has a very extensive literature, both theoretical and experimental: a review has been made by Di Prima & Swinney (1981). Alongside Bénard convection, Couette flow has provided an arena where new ideas and methods about stability, nonlinear behaviour and the approach to turbulence can be tested.

The question that we want to address in this paper is the following: what happens if the liquid contained between the two rotating cylinders is superfluid helium (helium II) instead of an ordinary classical liquid. Little is known about this 'superfluid Couette' problem compared to the vast literature existing about its classical counterpart.

From the experimental point of view, the superfluid case presents difficulties due to the low-temperature environment and the lack of flow visualization, but these are not insuperable problems. The major difference lies in the fact that we can study the classical Couette case on the sound ground of the Navier–Stokes equations, while we do not know with the same confidence the equations of motion of helium II.

A severe test which we can perform on proposed fluid equations is the prediction of a hydrodynamical instability. The purpose of this paper is therefore to study whether the equations which are believed to describe the motion of helium II can predict the onset of a secondary flow in the very simple Couette geometry.

Some experiments which are relevant to the study of this instability have been performed in the past and have been recently reviewed by Donnelly & LaMar (1988). They showed that the stability of the flow of helium is very different from the stability of a classical fluid. However no very clear picture emerges from the experiments. This is partly because the experiments have been performed at a variety of different temperatures, and the physical properties of helium II vary dramatically with temperature. What is clearly missing is a theoretical framework to help in interpreting the data of different experiments. It is the aim of our investigation to get such a framework, so as to understand the past experiments and to suggest how a future helium Couette experiment should be planned.

Some of the early experiments were performed in order to measure the viscosity of liquid helium (the Couette apparatus is indeed a viscometer). Donnelly & LaMar remarked that nowadays there is an uncertainty of 30 % in the value of the viscosity near the temperature of the λ-transition. Our study should therefore also benefit the design of a modern viscometer.

2. Previous work

The first pioneering attempt to calculate the stability of helium II flowing between rotating concentric cylinders was made by Chandrasekhar & Donnelly (1957). They used Landau's two-fluid model, which describes helium II as a mixture of two fluids: a perfect inviscid component (the superfluid) and a viscous component (the normal fluid). Detailed descriptions of the two-fluid model can be found in general references on superfluidity such as the books of Khalatnikov (1965) and Donnelly (1967). Here it suffices to remark that the relative amounts of normal fluid and superfluid depends on the temperature T. At $T = 0$ the fraction of normal fluid is zero and helium is entirely superfluid; as we increase the temperature the fraction of normal fluid grows, until it becomes one at the transition temperature $T = 2.172$ K, called the λ-point (we always consider saturated vapour pressure): then there is no superfluid left and helium II has become helium I, an ordinary classical liquid, albeit cold. If helium II rotates then the two-fluid model requires modifications, because the superfluid develops quantized vortex lines. In a simply connected container rotating at constant angular velocity Ω these lines are aligned along the axis of rotation and form an ordered array of density $n = 2\Omega/\Gamma$, where Γ is the quantum of circulation (Planck's constant divided by the mass of the helium atom). The thermal excitations (phonons and rotons) which make up the normal fluid collide with the cores of the vortex lines. The vortices therefore introduce a coupling between the normal fluid and the superfluid, called mutual friction, which was being investigated at the time when Chandrasekhar & Donnelly wrote their paper.

The argument of Chandrasekhar & Donnelly proceeded as follows. First they reasoned that in the absence of this coupling the two fluids would move independently. On the one hand, without mutual friction, the superfluid would obey Rayleigh's stability criterion for a classical inviscid fluid. Let us suppose that, for example, the outer cylinder is at rest. The superfluid would be unstable for any rotation velocity of the inner cylinder. On the other hand, without the coupling, the normal fluid would behave like an ordinary viscous liquid and would become unstable only if the velocity of the inner cylinder were higher than a critical value, as in the classical Couette problem. Following these considerations, Chandrasekhar & Donnelly calculated the effect of the mutual friction coupling and found that the first onset (the instability of the superfluid) is raised from zero to a finite rotation velocity of the inner cylinder.

In 1963 Mamaladze & Matinyan remarked that a new physical effect should be taken into account: the vortex lines have tension and thus they should provide a restoring force for a displaced element of fluid. They ignored the normal fluid, and thus the mutual friction, restricting their attention to the pure superfluid (helium II at zero temperature). They found that the instability occurs at non-zero rotation velocity of the inner cylinder, as did Chandrasekhar & Donnelly, but for a different reason (vortex tension as against mutual friction).

The modern description of the flow of helium II in the presence of vortex lines at non-zero temperature is given by the so-called Hall–Vinen–Bekharevich–

Khalatnikov equations (HVBK) (Hall & Vinen 1954; Bekharevich & Khalatnikov 1961; Hills & Roberts 1977). They are a generalization of Landau's two-fluid model to include mutual friction as well as vortex tension. The mutual friction force consists of a parallel and a transverse part, with coupling coefficients which are well known nowadays (Swanson *et al.* 1987); the calculation of Chandrasekhar & Donnelly contained only the parallel force. In 1974 Snyder briefly reported the preliminary results of trying to use the HVBK equations to predict the onset of instabilities in his helium Couette experiment. He computed a single stability point, which did not compare well with the experimental data. He did not follow up his work.

To conclude this summary of previous theoretical work on the subject, we finally note that in a recent paper (Barenghi & Jones 1987) we have found that the calculation of Mamaladze & Matinyan was unfortunately not correct. Our revised and extended calculation for the case of helium at zero temperature shows how the Rayleigh criterion is modified by the presence of the vortex lines, i.e. how the pure superfluid differs from the classical inviscid fluid. We shall come back to this in §6.

A complete review of the experimental work on the subject is already contained in the recent paper by Donnelly & LaMar and we do not need to summarize it here. In §6 we shall discuss in detail some selected experiments which, in view of our calculation, highlight the features of helium Couette flow. For the moment we simply remark that two different experimental techniques have been used. In the first method one measures the torque which the fluid, driven by one cylinder in motion, transmits to the other cylinder at rest. At small velocity the torque gives the value of the viscosity of the fluid; when the velocity of the moving cylinder reaches a critical value the linear relation between torque and angular velocity shows a break. Subsequent bifurcations in the flow pattern will, in general, give rise to further breaks in the torque–angular velocity curve. In the other method one makes use of second-sound waves. Second sound is attenuated in the bulk helium, but an extra attenuation is caused by the presence of the vortex lines which are set in a direction perpendicular to the direction of sound propagation. Thus second sound gives information about both the magnitude and the direction of the vorticity. However second sound does not exist above the λ-point, so this method can be used only in helium II, while the torque method can be used in both helium I and helium II. In the second-sound method, angular velocity is plotted against the attenuation factor, and breaks in the curve are interpreted as transitions in the flow.

Both these techniques suffer from the difficulty that it is not always clear which break in the curves corresponds to which transition in the flow. As we see later, it is fairly clear that in some experiments the breaks in the curves have not been interpreted correctly.

3. The equations of motion

Let R_1 and R_2 be the radii of the inner and outer cylinder, rotating at constant angular velocities Ω_1 and Ω_2 respectively. In our theory we make the assumption that the cylinders have infinite length, while in the real apparatus the length of the cylinders is h. This assumption is discussed below. Liquid helium is contained in the gap of width $\delta = R_1 - R_2$ between the cylinders. The total helium density is $\rho = \rho_n + \rho_s$, where ρ_s is the superfluid density and ρ_n is the normal fluid density (we use the subscripts s and n for the superfluid and the normal fluid and follow the notation introduced by Chandrasekhar & Donnelly 1957). Let p, S and T be the pressure, specific entropy and temperature. The viscosity of the normal fluid is η_n and its kinematic viscosity is $\nu_n = \eta_n/\rho_n$. The normal fluid and superfluid velocities $\boldsymbol{v}_n$ and $\boldsymbol{v}_s$ are macroscopic fields averaged over a region containing many vortices. The vortices have elastic properties, a fact demonstrated experimentally by the observation of vortex waves (see e.g. Glaberson & Donnelly 1986). The vortex-line tension parameter is $\nu_s = (\Gamma/4\pi)\log(b/a)$. In this expression the quantities a and b are the limits of integration of the kinetic energy of the superfluid around a vortex line: therefore for a we take the vortex core radius and for b we take the distance between vortices, $n^{-\frac{1}{2}}$, where n is the number of vortex lines per square centimetre. Note that ν_s has the dimension of a kinematic viscosity and in the situations in which we are interested ν_s is of the order of ν_n or larger.

The mutual friction force is proportional to

$$\boldsymbol{F} = \tfrac{1}{2}B\hat{\boldsymbol{\omega}}_s \times (\boldsymbol{\omega}_s \times (\boldsymbol{v}_n - \boldsymbol{v}_s - \nu_s \operatorname{curl}\hat{\boldsymbol{\omega}}_s)) + \tfrac{1}{2}B'\boldsymbol{\omega}_s \times (\boldsymbol{v}_n - \boldsymbol{v}_s - \nu_s \operatorname{curl}\hat{\boldsymbol{\omega}}_s), \tag{1}$$

where $\boldsymbol{\omega}_s = \operatorname{curl}\boldsymbol{v}_s$ and $\hat{\boldsymbol{\omega}}_s = \boldsymbol{\omega}_s/|\boldsymbol{\omega}_s|$ is a unit vector. This form of the force was deduced by second-sound experiments in rotating helium. The parallel dissipative part, proportional to a coefficient B, gives rise to a contribution to the attenuation of second sound; the transverse part, proportional to another coefficient B', couples otherwise degenerate modes in a resonant cavity. The coefficients B and B' are known from experiments. The parameters a, B, B', ρ, ρ_s, ρ_n and η_n are assumed to be only functions of the temperature; their values are taken from the article on mutual friction by Barenghi, Donnelly & Vinen (1983).

Since the velocities in which we are interested are small we assume the incompressibility conditions

$$\operatorname{div}\boldsymbol{v}_n = 0, \tag{2}$$

$$\operatorname{div}\boldsymbol{v}_s = 0. \tag{3}$$

Following Hills & Roberts (1977) we can now write the HVBK equations, which describe the motion of helium in the presence of vortices:

$$\frac{\partial \boldsymbol{v}_n}{\partial t} + (\boldsymbol{v}_n \cdot \boldsymbol{\nabla})\,\boldsymbol{v}_n = -\boldsymbol{\nabla}\sigma_n + \nu_n \nabla^2 \boldsymbol{v}_n + \frac{\rho_s}{\rho}\boldsymbol{F}, \tag{4}$$

$$\frac{\partial \boldsymbol{v}_s}{\partial t} + (\boldsymbol{v}_s \cdot \boldsymbol{\nabla})\,\boldsymbol{v}_s = -\boldsymbol{\nabla}\sigma_s - \nu_s \boldsymbol{\omega}_s \times \operatorname{curl}\hat{\boldsymbol{\omega}}_s - \frac{\rho_n}{\rho}\boldsymbol{F}, \tag{5}$$

where we have defined

$$\sigma_s = U - TS + \frac{p}{\rho} - \frac{\rho_n}{2\rho}(\boldsymbol{v}_n - \boldsymbol{v}_s)^2 + \frac{\rho_s \nu_s\,|\boldsymbol{\omega}_s|}{\rho}$$

and

$$\sigma_n = U + \frac{\rho_s}{\rho_n}TS + \frac{p}{\rho} + \frac{\rho_s}{2\rho}(\boldsymbol{v}_n - \boldsymbol{v}_s)^2 + \frac{\rho_s \nu_s\,|\boldsymbol{\omega}_s|}{\rho}.$$

Here p is the pressure: U and S are the internal energy and entropy per unit mass, which we assume to be constant at a given temperature. Note the term $\boldsymbol{\omega}_s \times \operatorname{curl}\hat{\boldsymbol{\omega}}_s$, which gives the acceleration of the superfluid due to bent vortex lines. The equations (4) and (5) are derived on the assumption that the vortices do not have tight curvature (a phenomenon which happens for example in the turbulent flow of the superfluid: Donnelly & Swanson 1986) and this is consistent with the problem we want to study. We have also to assume that the timescales of the flow in which we are interested are longer than the timescale for a single vortex to nucleate. As at this stage we want to keep the problem simple, we neglect the effects of collective motion of the vortices, called Tkachenko waves (see e.g. Glaberson & Donnelly 1986), and the velocity dependence of B and B' (Swanson *et al.* 1987), both of which have been observed experimentally.

We also ignore other ingredients which have been discussed in the literature but for which there is not clear experimental evidence, such as the possibility of a third term of axial mutual friction (which, if it exists, has to be very small: Mathieu, Placais & Simon 1984) and of a mass density in the vortex cores, which has been considered in the modern derivation of the HVBK equations (Hills & Roberts 1977).

The equations (1)–(5) have three limits of physical interest:

(i) if $T \rightarrow 2.172$ K then $\rho_s \rightarrow 0$ and the normal fluid equation (4) describes the classical Couette problem (in particular the flow of helium I). This limit $T \rightarrow 2.172$ K has to be regarded as a formal one, for the purpose of comparison with the classical viscous case. The vortex core parameter a will grow and diverge as the λ-transition is approached. In consequence, the assumption that a is small compared to the intervortex spacing will no longer be valid and the HVBK equations will no longer hold. This divergence of a only occurs, however, at temperatures very close to the λ-point temperature (Glaberson & Donnelly 1986).

(ii) if $T \rightarrow 0$ K then $\rho_n \rightarrow 0$ and the superfluid equation (5) describes a pure superflow (helium II at $T = 0$) which we have studied in a previous paper (Barenghi & Jones 1987).

(iii) if $T \rightarrow 0$ K and the quantum of circulation $\Gamma \rightarrow 0$, equation (5) describes the ideal fluid of classical fluid mechanics.

Furthermore we have the boundary conditions. In the Couette geometry the natural coordinates are cylindrical (r, ϕ, z). The superfluid can slip at the cylinders but the normal fluid cannot, so

$$\left.\begin{aligned} v_{nr}(r = R_1) &= v_{nr}(r = R_2) = 0, \\ v_{n\phi}(r = R_1) = \Omega_1 R_1, &\quad v_{n\phi}(r = R_2) = \Omega_2 R_2, \\ v_{nz}(r = R_1) &= v_{nz}(r = R_2) = 0, \\ v_{sr}(r = R_1) &= v_{sr}(r = R_2) = 0. \end{aligned}\right\} \tag{6}$$

4. The Couette flow solution

We have now to find the stationary solution that corresponds to Couette flow. We seek a steady axisymmetric solution of (1)–(5) of the form

$$\boldsymbol{v}_n = (0, v_n(r), 0), \quad \boldsymbol{v}_s = (0, v_s(r), 0). \tag{7}$$

The superfluid vorticity is in the z-direction, so that $\hat{\boldsymbol{\omega}}_s = \hat{\boldsymbol{z}}$. The mutual friction therefore has the form

$$\boldsymbol{F} = \tfrac{1}{2}B\hat{\boldsymbol{\phi}}(v_s - v_n)\frac{1}{r}\frac{d}{dr}(rv_s) + \tfrac{1}{2}B'\hat{\boldsymbol{r}}(v_s - v_n)\frac{1}{r}\frac{d}{dr}(rv_s). \tag{8}$$

From the ϕ-component of the superfluid equation (5), we have

$$(v_s - v_n)\frac{1}{r}\frac{d}{dr}(rv_s) = 0, \tag{9}$$

since the state we are seeking is independent of ϕ. We have two alternatives: either $v_s = K/r$ and there is no superfluid vorticity in the gap between the cylinders, or $v_s = v_n$. In either case (8) implies that $\boldsymbol{F} = 0$, so the ϕ-component of the normal fluid equation (4) leads to

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}r\frac{\mathrm{d}v_n}{\mathrm{d}r} - v_n/r^2 = 0, \tag{10}$$

with solution $v_n = Ar + C/r$; the constants A and C are determined by the boundary conditions at $r = R_1$ and R_2 to give

$$A = \frac{R_2^2\Omega_2 - R_1^2\Omega_1}{R_2^2 - R_1^2}, \quad C = \frac{R_1^2 R_2^2(\Omega_1 - \Omega_2)}{R_2^2 - R_1^2}, \tag{11}$$

as in classical Couette flow. Note that the vorticity associated with this flow is uniform between the cylinders and has value $2A$. On subtracting the radial components of (4) and (5) we obtain

$$\frac{\mathrm{d}T}{\mathrm{d}r} = \frac{\rho_n}{\rho S}(v_n - v_s)\left\{\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rv_n) - r\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{v_s}{r}\right)\right\}. \tag{12}$$

We now derive an important distinction between the alternative types of solution admitted by the HVBK equations: if $v_n = v_s$ an isothermal solution exists, but if there is no superfluid vorticity between the cylinders T cannot be constant.

The state we consider in this paper is the solution

$$v_s = v_n. \tag{13}$$

This implies that there are superfluid vortices in the gap, and that they have a uniform density given by $n = 2|A|/\Gamma$. If only the inner cylinder rotates then $A < 0$ and the vortices are antiparallel to the direction of rotation; on the other hand if only the outer cylinder rotates $A > 0$, and the vortices are parallel to the rotation.

There is some experimental evidence that the superfluid velocity is given at least approximately by (13) in the Couette regime (Donnelly & LaMar 1988). For rotation of either outer or inner cylinder Bendt (1967) found that the radial attenuation of second sound was proportional to the vorticity $|2A|$ of the Couette state, until the rotation is sufficiently high for an instability to result. This result is consistent with the vortices having a density proportional to $|A|$. Swanson & Donnelly (1987) argued that the first row of vortices appear (for narrow gaps) when

$$|A| > \Gamma \log(2\delta/\pi a)/\pi\delta^2 \tag{14}$$

and they present experimental evidence to support this view. The value of Ω corresponding to this threshold for the first entry of vortices is significantly below the value at which we expect instability. While this evidence supports (13), there are some uncertainties which should be borne in mind. The HVBK equations are based on an approximation in which the intervortex spacing is small compared with the lengthscale of the apparatus, which in this problem means the gap width δ. For some of the experiments this condition is satisfied reasonably well, but in others there may be only two or three rows of quantized vortices across the gap when instability occurs. As it is difficult to say how well the continuum approximation performs in these circumstances, it is important to bear in mind how many rows of vortices are present when comparing theory and experiment.

Another major problem is the uncertainty about the effect of the endwalls. In classical Couette flow the endwalls give rise to a meridional circulation even at slow flow; in consequence, when end effects are taken into account the transition to Taylor vortices is not a sharp bifurcation but a transition over a range of Reynolds numbers. The same behaviour must be expected here: the experiments are normally performed with fixed endwalls, and the normal fluid velocity must vanish there. The precise nature of the boundary conditions satisfied by the superfluid is not known. Experiments on the flow between oscillating disks suggested to Hall (1960) that the vortices are not necessarily pinned to the endwalls, and some slipping may occur. The amount of slipping may depend on the roughness of the surface.

In view of these uncertainties, it seems sensible to consider the infinite-cylinder theory first, and to relate to experiments where the length of the cylinders is large, and the wavelength of the instability is of the same order as the gap width.

5. The linearized perturbation equations

We consider a small perturbation of the Couette state $\boldsymbol{v}_{\mathrm{n}}^0, \boldsymbol{v}_{\mathrm{s}}^0$:

$$\boldsymbol{v}_{\mathrm{n}} \to \boldsymbol{v}_{\mathrm{n}}^0 + \boldsymbol{v}_{\mathrm{n}}, \quad \sigma_{\mathrm{n}} \to \sigma_{\mathrm{n}}^0 + \sigma_{\mathrm{n}},$$

$$\boldsymbol{v}_{\mathrm{s}} \to \boldsymbol{v}_{\mathrm{s}}^0 + \boldsymbol{v}_{\mathrm{s}}, \quad \sigma_{\mathrm{s}} \to \sigma_{\mathrm{s}}^0 + \sigma_{\mathrm{s}}.$$

If we linearize (1)–(5) and neglect terms that are of second or higher order in the perturbations, we obtain the following equations:

$$\frac{\partial}{\partial t} v_{\mathrm{n}r} = \mathrm{L}_r(\boldsymbol{v}_{\mathrm{n}}) - \frac{\partial \sigma_{\mathrm{n}}}{\partial r} + \nu_{\mathrm{n}}(\nabla^2 \boldsymbol{v}_{\mathrm{n}})_r + \frac{\rho_{\mathrm{s}}}{\rho} f_r, \tag{15}$$

$$\frac{\partial}{\partial t} v_{\mathrm{n}\phi} = \mathrm{L}_\phi(\boldsymbol{v}_{\mathrm{n}}) - \frac{1}{r}\frac{\partial \sigma_{\mathrm{n}}}{\partial \phi} + \nu_{\mathrm{n}}(\nabla^2 \boldsymbol{v}_{\mathrm{n}})_\phi + \frac{\rho_{\mathrm{s}}}{\rho} f_\phi, \tag{16}$$

$$\frac{\partial}{\partial t} v_{\mathrm{n}z} = \mathrm{L}_z(\boldsymbol{v}_{\mathrm{n}}) - \frac{\partial \sigma_{\mathrm{n}}}{\partial z} + \nu_{\mathrm{n}}(\nabla^2 \boldsymbol{v}_{\mathrm{n}})_z, \tag{17}$$

$$\frac{\partial}{\partial t} v_{\mathrm{s}r} = \mathrm{L}_r(\boldsymbol{v}_{\mathrm{s}}) - \frac{\partial \sigma_{\mathrm{s}}}{\partial r} + \nu_{\mathrm{s}} \frac{A}{|A|}\frac{\partial}{\partial z}\omega_{\mathrm{s}r} - \frac{\rho_{\mathrm{n}}}{\rho} f_r, \tag{18}$$

$$\frac{\partial}{\partial t} v_{\mathrm{s}\phi} = \mathrm{L}_\phi(\boldsymbol{v}_{\mathrm{s}}) - \frac{1}{r}\frac{\partial \sigma_{\mathrm{s}}}{\partial \phi} + \nu_{\mathrm{s}} \frac{A}{|A|}\frac{\partial}{\partial z}\omega_{\mathrm{s}\phi} - \frac{\rho_{\mathrm{n}}}{\rho} f_\phi, \tag{19}$$

$$\frac{\partial}{\partial t} v_{\mathrm{s}z} = \mathrm{L}_z(\boldsymbol{v}_{\mathrm{s}}) - \frac{\partial \sigma_{\mathrm{s}}}{\partial z}, \tag{20}$$

$$\nabla \cdot \boldsymbol{v}_{\mathrm{n}} = 0, \tag{21}$$

$$\nabla \cdot \boldsymbol{v}_{\mathrm{s}} = 0. \tag{22}$$

where the operator $\mathbf{L}$ is defined by

$$\mathrm{L}_r(\boldsymbol{v}) = 2\frac{v_{\mathrm{n}}^0}{r} v_\phi - \frac{v_{\mathrm{n}}^0}{r}\frac{\partial}{\partial \phi} v_r,$$

$$\mathrm{L}_\phi(\boldsymbol{v}) = -2A v_r - \frac{v_{\mathrm{n}}^0}{r}\frac{\partial}{\partial \phi} v_\phi,$$

$$\mathrm{L}_z(\boldsymbol{v}) = -\frac{v_{\mathrm{n}}^0}{r}\frac{\partial}{\partial \phi} v_z,$$

and where the Couette velocity $\boldsymbol{v}_{\mathrm{n}}^{0} = (0, v_{\mathrm{n}}^{0}, 0)$, and we define the perturbed mutual friction components

$$f_r = -|A|\,B(v_{\mathrm{n}r}-v_{\mathrm{s}r})-AB'(v_{\mathrm{n}\phi}-v_{\mathrm{s}\phi})-\nu_{\mathrm{s}}\tfrac{1}{2}B\frac{\partial}{\partial z}\omega_{\mathrm{s}\phi}+\nu_{\mathrm{s}}\tfrac{1}{2}B'\frac{A}{|A|}\frac{\partial}{\partial z}\omega_{\mathrm{s}r},$$

$$f_\phi = -|A|\,B(v_{\mathrm{n}\phi}-v_{\mathrm{s}\phi})+AB'(v_{\mathrm{n}r}-v_{\mathrm{s}r})+\nu_{\mathrm{s}}\tfrac{1}{2}B\frac{\partial}{\partial z}\omega_{\mathrm{s}r}+\nu_{\mathrm{s}}\tfrac{1}{2}B'\frac{A}{|A|}\frac{\partial}{\partial z}\omega_{\mathrm{s}\phi}.$$

It is useful to make the variables of our problem non-dimensional. We define

$$x = \frac{r-R_1}{\delta}, \quad \zeta = \frac{z}{\delta}, \quad \tau = \frac{\nu_{\mathrm{n}}}{\delta^2 t}, \tag{23}$$

and
$$V_{f,c} = \frac{\nu_{\mathrm{n}} v_{f,c}}{\delta}, \quad \sigma_f' = \frac{\nu_{\mathrm{n}}^2 \sigma_f}{\delta^2}, \tag{24}$$

where $f = \mathrm{n,s}$ and $c = r, \phi, z$.

The non-dimensional velocities are the Reynolds numbers of the inner and outer cylinder, Re_1 and Re_2 respectively, defined as

$$Re_1 = \Omega_1 R_1 \delta/\nu_{\mathrm{n}}, \quad Re_2 = \Omega_2 R_2 \delta/\nu_{\mathrm{n}}. \tag{25}$$

When $Re_2 = 0$ we shall make use of another non-dimensional velocity, the Taylor number

$$Ta = \frac{2\Omega_1^2\delta^4}{\nu_{\mathrm{n}}^2}\frac{\eta^2}{(1-\eta^2)}, \tag{26}$$

which is often used in the literature on the classical Couette problem. Finally we define the temperature-dependent parameters

$$\beta = \frac{\nu_{\mathrm{s}}}{\nu_{\mathrm{n}}}, \quad \gamma_{\mathrm{s}} = \frac{\rho_{\mathrm{s}}}{\rho}\tfrac{1}{2}B, \quad \gamma_{\mathrm{s}}' = \frac{\rho_{\mathrm{s}}}{\rho}\tfrac{1}{2}B', \quad \gamma_{\mathrm{n}} = \frac{\rho_{\mathrm{n}}}{\rho}\tfrac{1}{2}B, \quad \gamma_{\mathrm{n}}' = \frac{\rho_{\mathrm{n}}}{\rho}\tfrac{1}{2}B'. \tag{27}$$

We then assume that the perturbations have the dependence $\exp(\mathrm{i}m\phi+\mathrm{i}k\zeta+\mathrm{i}p\tau)$. Since we shall eliminate $v_{\mathrm{n}z}$ and $v_{\mathrm{s}z}$ from (15)–(22), the boundary conditions at $x = 0$ and $x = 1$ are

$$V_{\mathrm{n}r} = 0, \quad V_{\mathrm{n}\phi} = 0, \quad V_{\mathrm{s}r} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}x}V_{\mathrm{n}r} = 0. \tag{28}$$

The system of equations (15)–(22) is of eighth order. For comparison note that the classical Couette problem and the pure superfluid problem are respectively of sixth and second order.

The space of parameters is large: we have to assign the radius ratio $\eta = R_1/R_2$, the non-dimensional wavenumber k and the azimuthal wavenumber m of the perturbation, the Reynolds numbers Re_1 and Re_2 (if $Re_2 = 0$ we can also use the Taylor number Ta) and the parameters β, γ_{s}, γ_{s}', γ_{n} and γ_{n}' which are temperature dependent.

A comparison with a given experimental situation requires also the value of the gap δ. Note how β is obtained: ν_{s} has a logarithmic dependence on the ratio between the intervortex spacing b and the vortex core parameter a, which is determined by the temperature, T; b is found from δ, Re_1, Re_2 and ν_{n} (determined by T). Thus ν_{s} depends weakly on δ, and then we have $\beta = \nu_{\mathrm{s}}/\nu_{\mathrm{n}}$. We conclude that the eigenvalue problem is specified by the seven parameters η, Re_1, Re_2, k, m, T and δ. The sign of the imaginary part of the eigenvalue p determines the stability of the flow: if $\mathrm{Im}\,(p) < 0$ the flow is linearly stable.

The eigenvalue problem is solved numerically. The method of solution is given in the Appendix.

6. Results and comparison with experiments

Before discussing the stability of the flow of helium II at non-zero temperature, it is useful to summarize what is known about the following two limits:

(*a*) Pure superfluid limit $T \rightarrow 0$ K. We have studied this case in our previous paper (Barenghi & Jones 1987). Both for situations in which the inner cylinder only rotates, or the outer cylinder only rotates, we have found that the non-axisymmetric modes ($m \neq 0$) are unstable at long wavelength ($k \rightarrow 0$) and the critical velocity of rotation of the inner or outer cylinder tends to zero as $k \rightarrow 0$. In the case of rotation of the inner cylinder in the limit of narrow gap we have shown analytically that the $m = 0$ instability sets in at velocity Ω given by the root of

$$2\Omega - \nu_s k^2 = 4\Omega \left(\frac{\nu_s(1-\eta)}{\Omega \delta^2} \right)^{\frac{1}{2}}. \tag{29}$$

(*b*) Pure normal fluid limit, $T \rightarrow 2.172$ K. This is the Couette problem for a classical liquid. If the inner cylinder rotates at constant angular velocity Ω_1 and the outer cylinder is at rest, it is known that the Couette azimuthal flow is stable if Ω_1 is less than a critical velocity Ω_c. Above that critical velocity, modes with $m = 0$ become unstable and the flow breaks into a system of Taylor vortices. These vortices have axial wavelength corresponding to approximately square cells, $k \approx \pi$ being the critical wavenumber. If the outer cylinder rotates at constant angular velocity Ω_2 and the inner cylinder is at rest the flow is stable.

Let us now consider the general situation in which $0 < T < 2.172$ K. First we consider the case in which only the inner cylinder rotates. To start with, we concentrate our attention on the experiments of Donnelly (1959). His apparatus had a radius ratio $\eta = 0.95$ and the gap was $\delta = 0.1$ cm. The measurements were done by looking at the breaks in the relation between the Reynolds number and the torque induced on the outer stationary cylinder by the liquid helium, driven by the inner rotating cylinder. The experiment was repeated at three temperatures $T = 2.1$, 1.5 and 1.35 K.

At $T = 2.1$ K a first break was observed at $\Omega_1 = \Omega_c = 0.177$ rad/s; a second break was seen at $\Omega_1 = 0.213$ rad/s. In figure 1 we show the results of our theory, with the stability boundaries relative to this experiment, which we have computed for the modes $m = 0$, m = 1 and $m = 2$. In this graph the flow is stable below the stability boundary of a given mode and unstable above it; the minimum of the lowest curve determines the critical velocity: at this velocity an infinitesimal perturbation of wavelength $2\pi/k$ corresponding to the critical wavenumber k can grow exponentially and destabilize the flow. Figure 1 shows that the mode $m = 0$ is the first to become unstable: the critical Taylor number corresponds to a velocity of $\Omega_c = 0.181$ rad/s, in good agreement ($\approx 2\,\%$) with the observed first break.

The wavenumber of the instability is $k \approx 0.8$: since Donnelly's apparatus had height $h = 5$ cm, this implies that there were around 12 cells (6 cell pairs) in his experiment. Experience with classical Couette flow suggests this should be enough to get good agreement with the critical Taylor number. Note, however, that the cells are

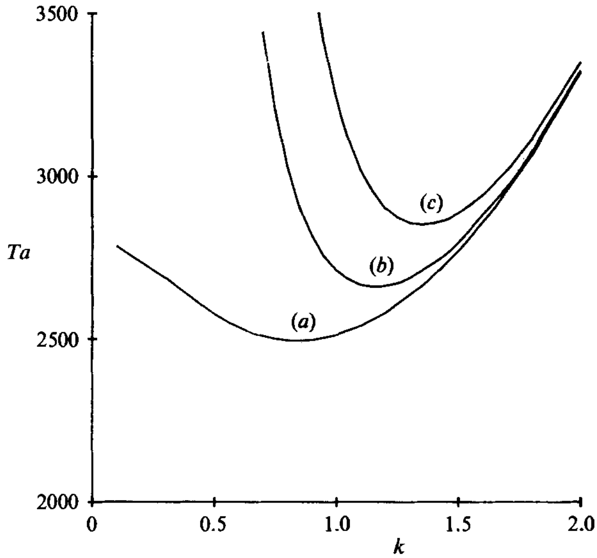

FIGURE 1. The stability boundaries at $\eta = 0.95, \delta = 0.1$ cm and $T = 2.1$ K for the modes: (a) $m = 0$, (b) 1, (c) 2. The geometry is as in the experiment of Donnelly (1959).

far from square: they are nearly 4 times as long as they are wide. We note also that at the critical velocity there are around 6 rows of vortices in the gap, which should be enough for our continuum approximation to be valid. Finally we remark that the velocity for the entry of vortices in Donnelly's apparatus is $\Omega^* = 0.047$ rad/s according to Swanson & Donnelly's theory: thus we have $\Omega^* \ll \Omega_c$ as one requires for consistency.

In the experiment no break was observed at $\Omega^* = 0.047$ rad/s: in fact the superfluid slips at the boundaries and the torque method should not be sensitive to the appearance of a row of vortices. We conclude that the Swanson & Donnelly theory and our theory provide us with a consistent picture of the experiment: the vortex array was created at $\Omega_1 \approx 0.047$ rad/s and became unstable at $\Omega_1 \approx 0.18$ rad/s.

Since the aim of our theory is to study the effects of both the vortex elasticity and the mutual friction on the stability of helium II, it is instructive to see what happens to the stability boundary of figure 1 if these ingredients are modified. If the mutual friction force is absent but the vortex elasticity is there, then the two fluids decouple and the superfluid is unstable for $\Omega_1 \rightarrow 0$ and $k \rightarrow 0$. Figure 2 shows what happens in the other case in which we keep the mutual friction, but we modify the vortex tension parameter from its full value (curve a) to half of it (curve b) to $\nu_s = 0$ (curve c): the critical velocity becomes 0.09 rad/s, much lower than the experimental data. Comparison of these curves shows that the vortex tension exerts a powerful influence even at $T = 2.1$ K, quite close to the λ-point, when the superfluid fraction is comparatively small. Perhaps even more significant than the increase in critical Taylor number is the increase in axial wavelength: with $\nu_s = 0$ the critical wavelength is close to the square-cell value, but with the full value of ν_s the axial wavelength is increased by a factor 4. Curve (d) on the same graph is the stability boundary calculated by using the analytical Chandrasekhar & Donnelly theory; the small difference between the curves (c) and (d) is probably because their theory assumes the limit $\eta \rightarrow 1$ and neglects the transverse part of the mutual friction.

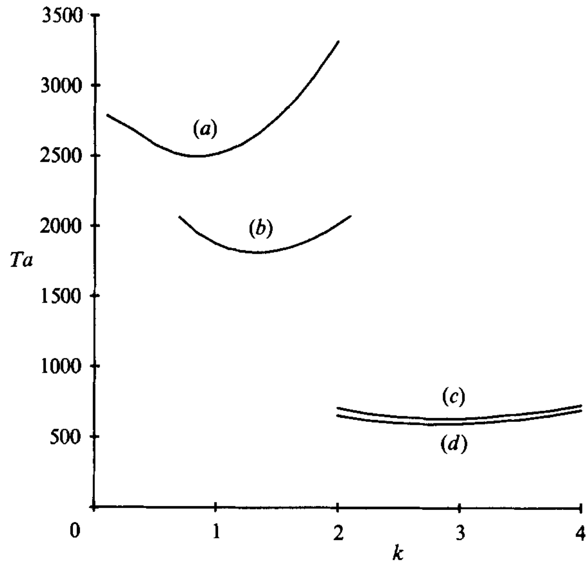

FIGURE 2. The effect of the vortex tension: the parameter values are as in figure 1. (*a*) The stability boundary of the $m = 0$ mode of figure 1; (*b*) the same but ν_s is arbitrary halved; (*c*) the same but $\nu_s = 0$, (*d*) the Chandrasekhar & Donnelly (1957) theory.

In their pioneering work Chandrasekhar & Donnelly claimed that two instabilities predicted by a linear theory should be observed, one being a raised Rayleigh instability of the superfluid and one being a shifted Taylor–Couette instability of the normal fluid. This led Donnelly to speculate that the second break that he observed at 0.213 rad/s had to do with the normal fluid. Measurements of the second and successive breaks have been done in other experiments, and some comments are thus necessary. At the first transition the mode that becomes unstable grows exponentially until the nonlinear terms determine a new stationary state. The second observed transition is an instability of this new state. This state is not the Couette state, however, so we cannot predict its instability from a linear perturbation of the Couette state, the approach used in both our calculation and Chandrasekhar & Donnelly's. However, in the classical Couette problem it is known that, if the gap is narrow, the second bifurcation (the transition from Taylor vortices to wavy Taylor vortices) appears at a velocity that is not much higher than the velocity of the first onset (Jones 1985*a*), and the linear perturbation theory of the Couette state is still able to predict approximately the second critical velocity. In the experiment of Donnelly the gap is narrow and the second observed critical Reynolds number is 20% higher than the first one; so we suspect, but we cannot really be certain, that the second onset has to do with the $m = 1$ mode of figure 1 which is predicted to onset at 3% higher Reynolds number than the $m = 0$ mode.

From the following considerations, however, it will become apparent that too little is known about the first onset of the helium Couette problem to worry much about the instabilities at higher velocities.

We study now what happens to the stability boundary of axisymmetric disturbances if the temperature changes. Figure 3 shows the results with the geometry of Donnelly's apparatus at different temperatures, $T = 2.16$, 2.1, 2.05 and 1.5 K, together with the curve of the classical Couette problem (at the same radius ratio), which describes helium just above the λ-point at $T = 2.172$ K. One can see

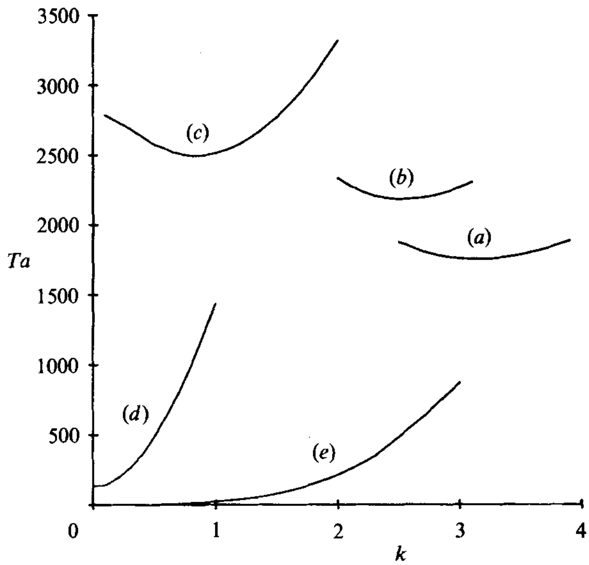

FIGURE 3. The effect of changing the temperature on the stability boundary of the $m = 0$ mode of figure 1. (*a*) The classical Couette case; (*b*) $T = 2.16$ K, (*c*) 2.1 K; (*d*) 2.05 K, (*e*) 1.5 K. Only the bottom part of each stability boundary is plotted for clarity.

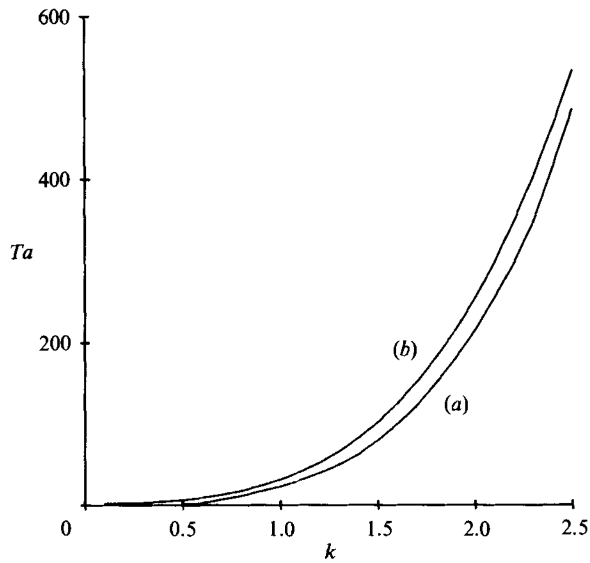

FIGURE 4. (*a*) The stability boundary for the $m = 0$ mode for Donnelly's experiment $\eta = 0.95$, $\delta = 0.1$ cm and $T = 1.5$ K. (*b*) The stability boundary for a pure superflow at $\eta = 0.95$ obtained from equation (29). The Taylor number was obtained from ν_n at $T = 1.5$ K.

that the boundary at $T = 2.16$ K is close to the classical curve, and indeed one should here recover the classical limit. As the temperature decreases the boundary moves to the left and the critical velocity changes, but eventually the curve loses its minimum and the instability sets in first at long wavelength $k \rightarrow 0$. This is indeed the behaviour of a pure superflow at $T = 0$: figure 4 shows that the curve corresponding to $T = 1.5$ K is close to the curve computed from the approximate expression (29) of a pure superfluid in the narrow-gap limit. Note that at $T = 1.5$ K $\rho_s/\rho = 0.88$, i.e. helium is

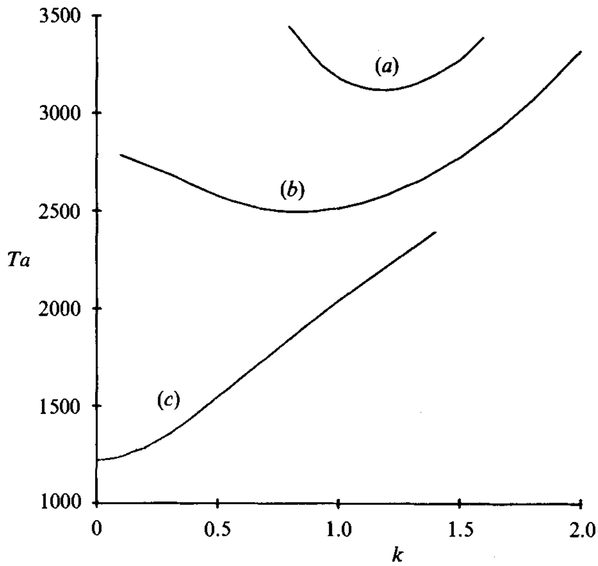

FIGURE 5. The stability boundary of the $m = 0$ mode at $T = 2.1$ K and $\delta = 0.1$ cm for different radius ratios η. (a) $\eta = 0.975$, (b) 0.95, (c) 0.925.

almost entirely superfluid. We conclude that we have thus recovered the expected results in the two limits of physical interest $T \rightarrow 2.172$ K and $T \rightarrow 0$.

Let us go back now to figure 3 which has important consequences for all helium Couette experiments. At low temperature the stability is governed by long wavelengths: if these are of the order of the height of the apparatus then the experiment is dominated by end effects. One does not need to reach a very low temperature for this to happen, since ρ_s/ρ is already about $\frac{1}{2}$ at $T = 1.95$ K and ν_s is generally greater than ν_n. This tendency of the stability curve to lose its minimum and to predict instabilities at $Ta \rightarrow 0$ is stronger if the radius ratio decreases: in figure 5 we plot the computed stability boundaries at $T = 2.1$ K for $\eta = 0.975, 0.95$ (which was Donnelly's value) and 0.925. The same features are seen in the case of the pure superflow (see figure 6).

In conclusion our model shows that the stability curve loses its minimum if the temperature is too low or the gap is too wide and assumes the form $Ta \rightarrow 0$ as $k \rightarrow 0$. In these cases we cannot really make any clear prediction: the finite height of the apparatus puts a lower bound on the possible wavenumbers k, and if the Taylor number is too small our continuum model fails since there are not enough vortices in the gap. We are thus unable to discuss quantitatively the other experiments of Donnelly at the lower temperatures $T = 1.5$ and 1.35 K, and the experiments of other authors reviewed by Donnelly & LaMar.

Let us now consider the case in which the outer cylinder rotates and $\Omega_1 = 0$. An early experiment by Heikkila & Hollis Hallet (1955) has shown that the Couette flow of helium becomes unstable if Ω_2 is larger than a critical velocity, in marked contrast to the behaviour of the classical Couette flow which is theoretically stable for all Reynolds numbers. The value of the viscosity, which Heikkila & Hollis Hallet found by looking at the torque induced on the inner cylinder for $\Omega_2 \rightarrow 0$, agrees with modern accepted values, so the observed instability is likely to be real.

In our previous work on the stability of a pure superflow we have found that rotation of the outer cylinder is unstable to non-axisymmetric modes ($m \neq 0$) and the

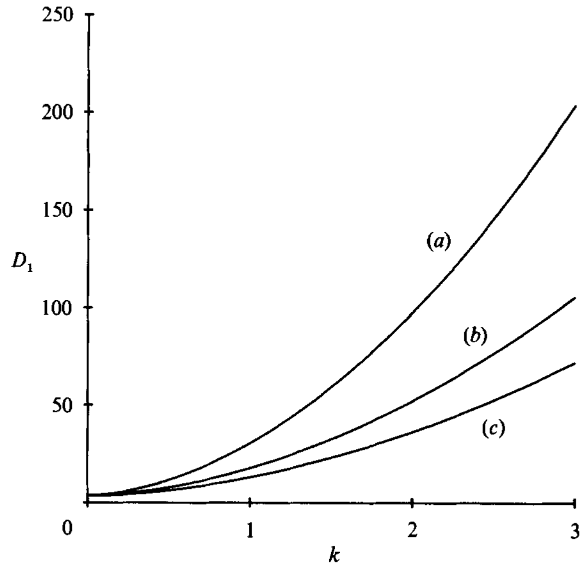

FIGURE 6. The stability boundary for the $m = 0$ mode of a superflow at different radius ratios η from equation (29). For a pure superflow it is convenient to define the non-dimensional velocity $D_1 = \Omega_1 R_1 \delta / \nu_s$. (a) $\eta = 0.975$, (b) 0.95, (c) 0.925.

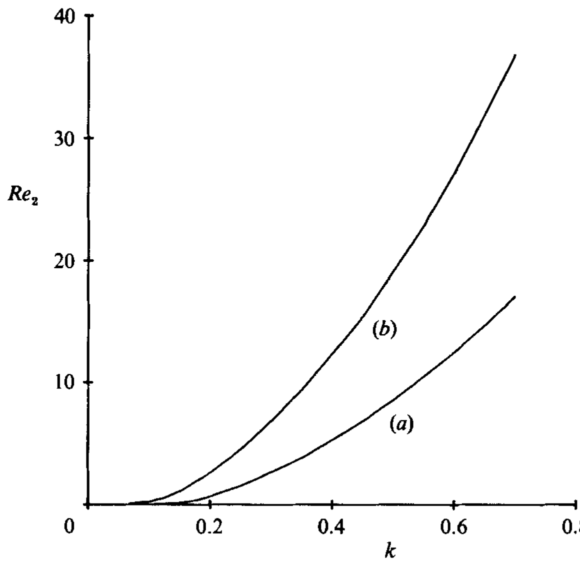

FIGURE 7. Rotation of the outer cylinder: the stability boundary of the $m = 1$ mode with $\eta = 0.95$, $\delta = 0.1$ cm at (a) $T = 1.308$ K, (b) 1.82 K; the geometry is as in the experiment of Heikkila & Hollis Hallet (1955).

stability boundaries are such that $\Omega_2 \rightarrow 0$ as $k \rightarrow 0$. This result was surprising, because the rotation of the outer cylinder and the elasticity of the vortices would have a stabilizing effect if taken separately, but it suggests that the remarkable difference between helium and a classical liquid when the outer cylinder rotates is due essentially to the vortex lines. One has only to see whether the results hold at non-zero temperature when mutual friction couples the unstable superfluid to the stable normal fluid. Figure 7 shows the stability boundaries for the $m = 1$ mode in the

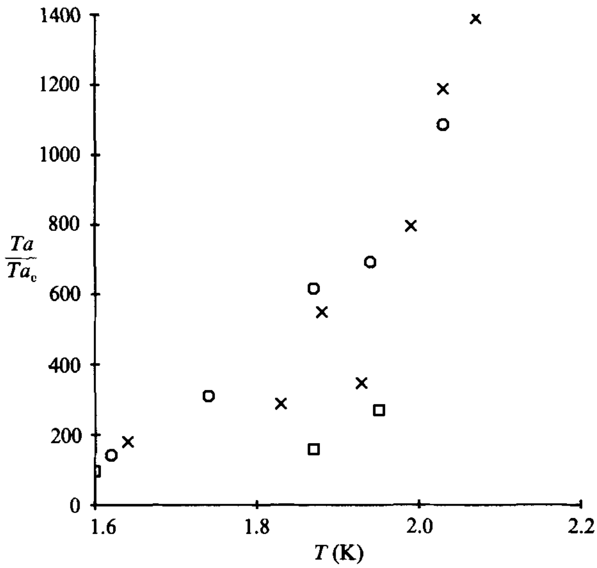

FIGURE 8. The data of Wolf *et al.* (1981). We plot Ta/Ta_c, where Ta is the Taylor number of the first observed onset in helium II and Ta_c is the critical Taylor number of the classical Couette case computed at the same η: $\times$, $\eta = 0.783$; $\bigcirc$, 0.8679; $\square$, 0.9434.

geometry of the experiment of Heikkila & Hollis Hallet, calculated at $T = 1.308$ and 1.82 K; there are other modes of higher m under these curves, as in the pure superflow case. However, we may conclude that the instability of the pure superflow for rotation of the outer cylinder exists at non-zero temperature. But again the fact that the critical velocities tend to zero as $k \rightarrow 0$ prevents us from making a quantitative comparison with the experiment. Since in their apparatus $h = 2.99$ cm and $\delta = 0.106$ cm the smallest meaningful value of k in figure 7 is 0.11.

In their recent paper Swanson & Donnelly (1987) have analysed this experiment and compared the measured critical velocity to their calculated velocity for the entry of vortices in the flow: they found that the values are very close and suggested that they are indeed the same. It is, however, not clear to us how the entry of vortices can be observed with the torque method. We can only conclude that the vortices become unstable as soon as they enter the flow, thus affecting the normal fluid component and hence the torque.

As in the case of the pure superflow, we find that the unstable eigenfunctions peak closer to the stationary inner cylinder as the outer cylinder moves faster, and that the frequency $\mathrm{Re}\,(p)$ of the perturbations is very small. If one compared radial second-sound resonances of different orders one could perhaps test this prediction that the vorticity is concentrated near the inner stationary cylinder. Azimuthal second-sound resonances are split by the Doppler effect: since the phase velocity $\mathrm{Re}\,(p)/m$ is small, one should observe a very small resonance splitting (on top of the one due to the base flow v_s^0), in contrast to the Doppler separation for rotation of the inner cylinder only (in that case $\mathrm{Re}\,(p)/m$ at the onset has a similar value to that found in the classical Couette case).

Whereas the experiments discussed above used the torque method, an alternative method of detecting transitions using second sound has been used in an experiment by Wolf *et al.* (1981): the results however disagree very substantially with the torque results of Donnelly. In the Wolf *et al.* experiment the outer cylinder was held stationary. Three radius ratios were used to study the first onset, $\eta = 0.783$, 0.8673 and 0.9434: the corresponding aspect ratio h/δ were 15.6, 25.6 and 60.0. We have reanalysed the data that Wolf *et al.* claim to be the first instability of the vortex array by plotting them in the form of Ta/Ta_c as a function of temperature T (see figure 8); here Ta is the critical Taylor number measured in helium, while Ta_c is the critical Taylor number for a classical fluid, which we have calculated at the same radius ratio used in the helium experiment: these classical critical Taylor numbers are well-known: see e.g. Di Prima & Swinney (1981)). For the three radius ratios used in these experiments Ta_c has the values 2032, 1877, and 1763 respectively. One expects that $Ta/Ta_c \rightarrow 1$ as $T \rightarrow 2.172$ K. On the contrary, Ta/Ta_c increases as T increases, and at the highest measured temperature it is more than three orders of magnitude bigger than what one expects. Even the lowest values of Ta/Ta_c found in this experiment are nearly two orders of magnitude above the typical values we have computed. This should be compared to $Ta/Ta_c \approx 1.4$ found in Donnelly's experiment at $T = 2.1$ K and $\eta = 0.95$. One may conjecture that perhaps what was observed in the Wolf *et al.* experiment was not the first instability but a succeeding one: the azimuthal resonances used in the experiment perhaps were not sensitive to long wavelengths along the axial direction, because the increase in length of the vortices, set normally to the azimuthal direction of propagation of second sound, was not sufficient to cause the extra attenuation to be detected.

It seems important to make sure that nothing as extraordinary as the discontinuity at the λ-point suggested in figure 8 really happens. Above the λ-point second sound does not exist, and so only the torque method can be used in both helium I and helium II to clarify this problem.

7. Discussion

The most important result emerging from this work is that the vortex tension profoundly affects the nature of the instability of Couette flow. In classical Couette flow, with large aspect ratios, the Taylor cells that develop far from the endwalls are hardly affected by the precise nature of the end conditions: in consequence, experiments give values of the critical Reynolds number that are in good agreement with infinite cylinder theory. When we are in the regime where vortex tension dominates, this happy situation no longer prevails. Non-axisymmetric modes with long axial wavelength are the first to become unstable, which means that the end conditions will significantly affect the onset of instability. One then has to worry about whether the superfluid vortices are pinned to the endwalls or can slip, something which may depend on the roughness of the endwall surfaces.

In view of this, it would seem that a future experiment on helium Couette flow should be designed towards exploring the transition region between the classical Couette regime at temperatures near the λ-point and the low-temperature vortex-tension-dominated regime. The results presented here indicate that the transition region exists only over a rather narrow temperature range $2.05\ \mathrm{K} < T < 2.172$ K; even at the comparatively high temperature of 2.05 K vortex tension has a strong influence. It is only in this transition region that critical wavelengths shorter than the height of the apparatus will be found, and hence the dependence on end effects eliminated. Naturally, these considerations strongly favour long aspect ratios. The results from varying the radius ratio suggest that this should be as close to one as is practical, as the favoured axial wavelength increases sharply as the radius ratio is lowered; there is a substantial difference between $\eta = 0.95$ and $\eta = 0.90$. The number of rows of vortices across the gap at the onset of instability also increases as $\eta \to 1$, so a radius ratio close to unity will improve the validity of the continuum approximation.

Previous experiments have been conducted over the much wider temperature range $1.3\ \mathrm{K} < T < 2.172\ \mathrm{K}$. In consequence, most of the experiments have been entirely in the vortex-dominated regime where the uncertainties are greatest. We believe that this is the principal reason why no very clear picture has emerged from these experiments. We are encouraged by the fact that Donnelly's experiment at 2.1 K, one of the very few to lie in the transition zone, and have suitable radius and aspect ratio, gives very good agreement with the theoretical results.

We have said that two methods (second sound and torque) have been used in helium Couette experiments. The measurement of the attenuation of second sound is in principle the best method because it allows direct detection of the vorticity: second sound is not attenuated along the direction of the vortex lines, so a combination of axial, radial and azimuthal resonances or pulses would give much information about the vortices. Moreover, different modes can probe different regions: the fundamental radial mode is attenuated mainly by the vortices in the middle of the gap, while higher harmonics probe the boundary regions.

Nevertheless, we note that in the experiments done so far, the sensitivity of the second-sound method for detecting the onset of instability is somewhat in question, and the results from the torque method have given results in better agreement with theory.

We are indebted to Professor R. J. Donnelly for having stimulated this work and for his hospitality at the University of Oregon. We also wish to thank Dr C. E. Swanson and Professor H. A. Snyder for helpful discussions. Dr C. F. Barenghi acknowledges support from SERC grant GR/D/30433.

Appendix

We give here a brief description of the numerical techniques we have used to obtain our results. After elimination of the pressures σ'_{n} and σ'_{s} and of the axial components of the velocities $V_{\mathrm{s}z}$ and $V_{\mathrm{n}z}$, the dimensionless form of the system (15)–(22) is reduced to four equations in $V_{\mathrm{n}r}$, $V_{\mathrm{n}\phi}$, $V_{\mathrm{s}r}$ and $V_{\mathrm{s}\phi}$:

$$\sum_{f=\mathrm{n,s}} \sum_{c=r,\phi} \mathrm{L}_{efc} V_{fc} = p \sum_{f=\mathrm{n,s}} \sum_{c=r,\phi} \mathrm{R}_{efc} V_{fc}, \quad e = 1 \quad \text{to } 4, \tag{A 1}$$

where L and R are operators of the form

$$\mathrm{L}_{efc} = \sum_{k=1}^{4} \mathrm{L}^{k}_{efc}(x) \frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}, \quad \text{and similarly for } \mathrm{R}_{efc}. \tag{A 2}$$

Experience with similar flow problems (Jones 1985*b*) shows that it is convenient to expand the velocities over truncated series of the modified Chebyshev polynomials $T^{*}_{j}(x)$, which are defined in the interval $0 \leqslant x \leqslant 1$,

$$\left.\begin{aligned} V_{\mathrm{n}r} &= \sum_{j=1}^{N(\mathrm{n}r)} a_{\mathrm{n}rj} T^{*}_{j-1}(x), \quad & V_{\mathrm{n}\phi} &= \sum_{j=1}^{N(\mathrm{n}\phi)} a_{\mathrm{n}\phi j} T^{*}_{j-1}(x), \\ V_{\mathrm{s}r} &= \sum_{j=1}^{N(\mathrm{s}r)} a_{\mathrm{s}rj} T^{*}_{j-1}(x), \quad & V_{\mathrm{s}\phi} &= \sum_{j=1}^{N(\mathrm{s}\phi)} a_{\mathrm{s}\phi j} T^{*}_{j-1}(x). \end{aligned}\right\} \tag{A 3}$$

We now define the collocation points $x_i, i = 1 \dots N$ which are the zeros of the polynomial $T^*_N(x)$. The four equations are evaluated at these points, and together with the eight boundary conditions we have

$$\sum_{f=\mathrm{n,s}} \sum_{c=r,\phi} \sum_{j=1}^{N(fc)} (\mathrm{L}_{efc} T^*_{j-1})(x_i) = p \sum_{f=\mathrm{n,s}} \sum_{c=r,\phi} \sum_{j=1}^{N(fc)} (\mathrm{R}_{efc} T^*_{j-1})(x_i)$$

for $e = 1, 2, 3, 4$ and $i = 1, \dots, N$. These equations can be written in matrix form as

$$\boldsymbol{A} \cdot \boldsymbol{X} = p \boldsymbol{B} \cdot \boldsymbol{X} \tag{A 4}$$

where the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ contain respectively the operators L and R, p is the eigenvalue and $\boldsymbol{X}$ is a vector of length

$$N(\mathrm{n}r) + N(\mathrm{n}\phi) + N(\mathrm{s}r) + N(\mathrm{s}\phi)$$

containing the unknown coefficients $a_{\mathrm{n}rj}, a_{\mathrm{n}\phi j}, a_{\mathrm{s}rj}, a_{\mathrm{s}\phi j}$ of the Chebyshev expansions of the velocities. We then use standard routines of the Numerical Algorithm Group Library (NAG) to invert the matrix $\boldsymbol{B}$ and to solve the eigenvalue equation

$$(\boldsymbol{B}^{-1}\boldsymbol{A})\boldsymbol{X} = p\boldsymbol{X}.$$

However, before doing so, care must be used to avoid columns or rows of zeros in $\boldsymbol{B}$. We deal with the boundary conditions by choosing $N(\mathrm{n}r) = N+4$, $N(\mathrm{n}\phi) = N+2$, $N(\mathrm{s}r) = N+2$ and $N(\mathrm{s}\phi) = N$. We eliminate the last terms $j = N+1$ to $N+4$ of the expansion of $V_{\mathrm{n}r}$, the terms $j = N+1$ to $N+2$ of the expansion of $V_{\mathrm{n}\phi}$ and the terms $j = N+1$ to $N+2$ of the expansion of $V_{\mathrm{s}r}$ by direct substitution using the boundary conditions. In this way $\boldsymbol{A}$ and $\boldsymbol{B}$ are complex square matrices of size $4N \times 4N$, the expansion of each velocity component has N terms and we can proceed by inverting $\boldsymbol{B}$ and finding the eigenvalue as stated above. The NAG routine returns then $4N$ eigenvalues: not all of them are meaningful and some are not accurate, because eigenfunctions which have many oscillations cannot be approximated well with a truncated Chebyshev series, and need more polynomials. How do we identify the good eigenvalues? First one notes that in the Chebyshev representation of a good smooth eigenfunction the first coefficients are much bigger than the last ones in the series, otherwise the function is not well approximated; secondly one can substitute a given eigenfunction and eigenvalue into the original equations and check that they are satisfied on a mesh of equally spaced points together with the boundary conditions. But the test which was always performed because it is very strict was the following: we go back to the original system (15)–(22) and derive eight equations of the first order, in a manner analogous to that used by Roberts (1965). The eigenvalue and eigenfunctions are then obtained using an orthonormalization method (Conte 1966), based on a fourth-order Runge–Kutta scheme, and using the eigenvalue found by means of the Chebyshev technique as the initial guess. In this way all eigenvalues have been confirmed by an independent method.

REFERENCES

Barenghi, C. F., Donnelly, R. J. & Vinen, W. F. 1983 Friction on quantized vortices in helium II. *J. Low Temp. Phys.* **52**, 189–247.

Barenghi, C. F. & Jones, C. A. 1987 On the stability of superfluid helium between rotating concentric cylinders. *Phys. Lett.* A **122**, 425–430.

Bekharevich, I. L. & Khalatnikov, I. M. 1961 Phenomenological derivation of the equations of motion in HeII. *Sov. Phys. JETP* **13**, 643–646.

BENDT, P. J. 1967 Attenuation of second sound of helium II between rotating cylinders. *Phys. Rev.* **153**, 280–284.

CHANDRASEKHAR, S. & DONNELLY, R. J. 1957 The hydrodynamic stability of helium II between rotating cylinders I. *Proc. R. Soc. Lond.* A**241**, 9–28.

CONTE, S. D. 1966 The numerical solution of linear boundary value problems. *SIAM Rev.* **8**, 309–321.

DI PRIMA, R. C. & SWINNEY, H. L. 1981 Instabilities and transition in flow between concentric rotating cylinders. In *Hydrodynamic Instabilities and Transition to Turbulence* (ed. H. L. Swinney & J. P. Gollub), pp. 139–180. Springer.

DONNELLY, R. J. 1967 *Experimental Superfluidity*. University of Chicago Press.

DONNELLY, R. J. 1959 Experiments on the hydrodynamic stability of helium II between rotating cylinders. *Phys. Rev. Lett.* **3**, 507–508.

DONNELLY, R. J. & LAMAR, M. M. 1988 Flow and stability of helium II between concentric cylinders. *J. Fluid Mech.* **186**, 163–198.

DONNELLY, R. J. & SWANSON, C. E. 1986 Quantum turbulence. *J. Fluid Mech.* **173**, 387–429.

GLABERSON, W. I. & DONNELLY, R. J. 1986 Structure, distribution and dynamics of vortices in helium II. In *Progress in Low Temperature Physics*, vol. IX (ed. D. F. Brewer), pp. 3–142. North-Holland.

HALL, H. E. 1960 The rotation of liquid helium II. *Phil. Mag. Suppl.* **9**, 89–146.

HALL, H. E. & VINEN, W. F. 1954 The rotation of liquid helium II. II: the theory of mutual friction in uniformly rotating helium II. *Proc. R. Soc. Lond.* A**238**, 215–234.

HEIKKILA, W. J. & HOLLIS HALLET, A. C. 1955 The viscosity of liquid helium II. *Can. J. Phys.* **33**, 420–435.

HILLS, R. N. & ROBERTS, P. H. 1977 Superfluid mechanics for a high density of vortex lines. *Arch. Rat. Mech. Anal.* **66**, 43–71.

JONES, C. A. 1985*a* The transition to wavy Taylor vortices. *J. Fluid Mech.* **157**, 135–162.

JONES, C. A. 1985*b* Numerical methods for the transition to wavy Taylor vortices. *J. Comp. Phys.* **61**, 321–344.

KHALATNIKOV, I. M. 1965 *An Introduction to the Theory of Superfluidity*. Benjamin.

MAMALADZE, Y. G. & MATINYAN, S. G. 1963 Stability of rotation of a superfluid liquid. *Sov. Phys. JETP* **17**, 1424–1425.

MATHIEU, P., PLACAIS, B. & SIMON, Y. 1984 Spatial distribution of vortices and anisotropy of mutual friction in rotating HeII. *Phys. Rev.* B**29**, 2489–2496.

ROBERTS, P. H. 1965 Appendix to 'Experiments on the stability of viscous flow between rotating cylinders' (by R. J. Donnelly & K. W. Schwarz). *Proc. R. Soc. Lond.* A**283**, 531–556.

SNYDER, H. A. 1974 Rotating Couette flow of superfluid helium. In *Proc. 13th Intl Conf. Low Temp. Phys. LT13*, vol. 1, pp. 283–287. Plenum.

SWANSON, C. E. & DONNELLY, R. J. 1987 Appearance of vortices in Taylor–Couette flow of helium II. *J. Low Temp. Phys.* **67**, 185–193.

SWANSON, C. E., WAGNER, W. T., BARENGHI, C. F. & DONNELLY, R. J. 1987 Calculation of frequency and velocity dependent mutual friction parameters in helium II. *J. Low Temp. Phys.* **66**, 263–276.

WOLF, P. L., PERRIN, B., HULIN, B. & ELLEAUME, J. P. 1981 Rotating Couette flow of helium II. *J. Low Temp. Phys.* **44**, 569–593.